Elementary proof of Picard little theorem

We present the proof of Picard little theorem by John L. Lewis in [1]. Here we only extract the essential part for the proof in the paper.

For any harmonic function u on $\mathbb{C}$, by Poisson formula, we have

$$u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{\pi}^{\pi}\left[\frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}u(a+Re^{i\theta})\,dt\right],\forall a\in\mathbb{C},\ 0\le r<R.$$

And thus, if $u\ge 0$ on $B(a,R)$

$$\frac{R-r}{R+r}u(a)\le u(a+re^{i\theta})\le\frac{R+r}{R-r}u(a) \tag{1}$$

by mean of mean value property of harmonic function. This inequality is called Harnack inequality. In particular, we have

$$\sup\{u(z):|z-a|<r\}\le 9\cdot\inf\{u(z):|z-a|<r\}$$

whenever $u\ge 0$ on $B(a,2r)$.

For simplicity, we denote $M(x,r)=\sup\{u(z):|z-x|<r\}$.

Lemma 0.1. *Let u be a harmonic function, $u(a)=0$, and $R>0$. Then there exists $r\in(0,R)$, $x_1\in B(a,2R)$, and a universal constant $c_1\ge 2$ such that $u(x_1)=0$ and*

$$c_1^{-1}M(a,R)\le M(x_1,10r)\le c_1M(x_1,r).$$

Proof. Let $\delta(x)=2R-|x-a|$. Put $E=\{x:u(x)=0\}\cap B(a,2R)$. Let $F=\overline{\cup_{x\in E}B(x,\delta(x)/100)}$. Set

$$\gamma=\sup\{M\left(x,\delta(x)/100\right)\ :x\in E\}.$$

Noted that $\gamma>0$ otherwise $u\equiv 0$ by maximum principle. Choose $x_1\in E$ such that

$$\gamma\le 2M(x_1,r)\quad\text{where}\ \ r=\delta(x_1)/100.$$

We finish the proof by showing that this x_1 and r satisfy our goal. First we have for $y\in B(x_1,20r)$,

$$\delta(x_1)\le 2\delta(y)\le 4\delta(x_1).$$

Pick a $x_2\in\bar{B}(x_1,10r)$ with

$$M(x_1,10r)\le 2u(x_2).$$

case 1: If $x_2\in F$,

$$M(x_1,10r)\le 2u(x_2)\le 2\gamma\le 4M(x_1,r).$$

case 1: If $x_2\notin F$,

Denote (a,b) to be the open line segment connecting a,b. Denote closed, half open line segment similarly. Since F is closed, there exists $z \in (x_1, x_2) \cap F$ such that $[x_2, z) \cap F = \phi$. For each $w \in [x_2, z)$, it contains a ball of radius $r/4$ on which $u \geq 0$. Otherwise, by continuity there exists a y such that $u(y) = 0$ with

$$|y - w| \leq \frac{r}{4} = \frac{\delta(x_1)}{400} < \frac{\delta(y)}{100}.$$

Thus $w \in F$ which contradicts with our choice of z.

Since $[x_2, z]$ can be covered by 80 balls of radius $r/8$, and u is nonnegative on each balls, apply Harnack inequality, we yield

$$M(x_1, 10r) \leq 2u(x_2) \leq 2 \cdot 9^{80} u(z) \leq 4 \cdot 9^{80} M(x_1, r).$$

For the left hand side, choose $x_3 \in \bar{B}(a, R)$ such that $2u(x_3) \geq M(a, R)$. Then argue in the case of $x_3 \in F$ or $x_3 \notin F$ as before. □

Theorem 0.2. *Little Picard theorem: A nonconstant entire function in the complex plane omits atmost one value.*

Proof. We prove by contradiction. Without loss of generality, we assume f omits 0 and 1. Put $u_1 = \log |f|$, $u_2 = \log |f - 1|$ which are harmonic in $\mathbb{C}$. It can be seen that all positive (or negative) harmonic functions are constant by letting $R \to \infty$ in (1). So we can choose $a \in \mathbb{C}$ such that $u(a) = 0$. Applying Lemma 0.1 to u_1, with $R = 2^j, j = 1, 2, ...$ to obtain a sequence $\{z_j\}, \{r_j\}$ with

1. $\lim_{j\to\infty} M(z_j, r_j) = +\infty$,
2. $M(z_j, 10r_j) \leq c_1 M(z_j, r_j)$,
3. $u_1(z_j) = 0$ for $j = 1, 2, ...$

Define

$$v_{i,j}(z) = u_i(z_j + 10r_j z)/M(z_j, 10r_j) \text{ on } B(0, 1),\ i = 1, 2,\ j \in \mathbb{N}.$$

By statement (2), we have a subsequent convergence of $v_{i,j} \to v_i$ uniform on any compact subset of $B(0, 1)$. And it satisfies

(*) $v_i(0) = 0$ for $i = 1, 2$.

(**) $v_1 = v_2$ on $\{x : v_1(x) > 0\} \cup \{x : v_2(x) > 0\} \neq \phi$.

(***) $\{x : v_1(x) < 0\} \cap \{x : v_2(x) < 0\} = \phi$.

Since $M(z_j, 10r_j) \leq c_1 M(z_j, r_j)$, there exists $x_j \in B(0, 1/2)$ such that

$$M(z_j, 10r_j) \leq c_1 M(z_j, r_j) = c_1 u_1(z_j + 10r_j x_j).$$

Taking limit implying $\{x : v_1(x) > 0\} \cup \{x : v_2(x) > 0\} \neq \phi$ (pass to subsequence if necessary). If $v_1(z) = c > 0$, for large j, $|f(z_j + 10r_j z)| >> 0$, thus $\log \frac{|f-1|}{|f|}$ is bounded implying $v_2(z) = v_1(z)$. So we have (**).

If $z \in \{x : v_1(x) < 0\} \cap \{x : v_2(x) < 0\}$, we have for some $c < 0$, for large j

$$\begin{cases} \log|f(z_j + 10r_j z)| < cM(z_j, 10r_j) \to -\infty, \\ \log|f(z_j + 10r_j z) - 1| < cM(z_j, 10r_j) \to -\infty \end{cases} \tag{2}$$

which is not possible. Thus (***) is verified.

Since v_i are real analytic, by identity theorem and (**), $v_1 \equiv v_2$. By (***), $v_1 \geq 0$. By maximum principle and (*), $v_1 \equiv 0$ which contradicts with (**). □

References

[1] J. Lewis, Picards theorem and Rickmans theorem by way of Harnacks inequality, Proc. AMS, 122 (1994) 199206